Indian Statistical Institute, Bangalore
M. Math II, Second Semester, 2022-23
Mid-semester Examination,
Special Topics - Quantitative Finance

21.02.23 **Maximum Score 75** **Duration: 3 Hours**

Students are allowed to use class notes and the book Stochastic Calculus for Finance II by Shreve.

1. (15) Consider two markets, $\mathcal{M}_1$ and $\mathcal{M}_2$ both with Ω as the set of possible scenarios and both with $A_1, A_2, \cdots, A_K$ as the set of freely traded assets. In market $\mathcal{M}_1$, the share price of asset A_j is S_j^0 at $t = 0$ and, in scenario ω, it is $S_j^1(\omega)$ at $t = 1$. In market $\mathcal{M}_2$, the corresponding prices are $\tilde{S}_j^0$ and $\tilde{S}_j^1(\omega)$ where

$$\tilde{S}_j^0 = S_j^0/S_1^0 \quad \text{and} \quad \tilde{S}_j^1(\omega) = S_j^1(\omega)/S_1^1(\omega).$$

Thus, share prices in market $\mathcal{M}_2$ are quoted in shares of A_1. Observe that in market $\mathcal{M}_2$, asset A_1 is riskless, with rate of return 0, provided that its share price remains positive under every market scenario.

(a) Assuming the price of A_1 is positive at time zero and at time one in every market scenario, show that $\mathcal{M}_1$ is arbitrage free iff $\mathcal{M}_2$ is arbitrage free.

(b) Suppse $\mathcal{M}_2$ has equilibrium measure $\tilde{\pi}(\omega)$. Show that the measure $\pi(\omega)$ given below is an equilibrium measure for $\mathcal{M}_1$.

$$\pi(\omega) = \tilde{\pi}(\omega)\frac{S_1^0}{S_1^1(\omega)}/\delta$$

where δ is the discount factor in $\mathcal{M}_1$ given by

$$\delta = \sum_{\omega} \tilde{\pi}(\omega)\frac{S_1^0}{S_1^1(\omega)}.$$

2. (10) Let $\theta(s) = W(s)I(s \le 1)$ where $W(s)$ is standard Brownian motion. Suppose we approximate $\theta(s)$ by a sequence of simple processes $\theta^{(n)}(s)$ defined as

$$\theta^{(n)}(s) = \sum_{k=0}^{2^n-1} \theta\left(\frac{k}{2^n}\right) I\left(\frac{k}{2^n} \le s < \frac{k+1}{2^n}\right).$$

Show that $\theta^{(n)}(s)$ approximates $\theta(s)$ in $\mathcal{H}_2$, that is,

$$\lim_{n\to\infty} \int_0^\infty E(\theta(s) - \theta^{(n)}(s))^2 ds = 0.$$

3. (25) Consider the Black-Scholes model under the risk-neutral measure with the risk-free rate $r = 0$.

   (a) Show that the conditional distribution of S_T given $S_t = S$ is same as the distribution of S_{T-t} when the initial price S_0 is S.

   (b) Let $C(S, t, K, T)$ denote the price of a call option with expiration T and strike T at time t with stock price $S_t = S$. Using part (a), show that

   $$C(S, t, K, T) = C(S, 0, K, T - t).$$

   Hence find a PDE for C where the partial derivatives are taken with respect to t and T.

   (c) For any constant a, show that

   $$C(aS, t, aK, T) = aC(S, t, K, T).$$

   Hence find a PDE for C where the partial derivatives are taken with respect to S and K.

4. (25) For the Black-Scholes model under risk neutral measure with $r > 0$, we obtained the price of an up and in barrier option with payoff 1 and barrier at $A(> S_0)$ as

   $$e^{-\theta^2/2} \int_0^\infty e^{-\theta\alpha}(e^{\theta x} + e^{-\theta x})e^{-(x+\alpha)^2/2T} \frac{1}{\sqrt{2\pi T}} dx.$$

   Here $\theta = -\frac{1}{\sigma}(r - \frac{\sigma^2}{2})$ and $\alpha = \frac{1}{\sigma}\ln(A/S_0)$.

   (a) Express the price in terms of the standard normal cdf.

   (b) Show that the event $M_T > A$ is identical to the event $\tau_A < T$, where M_T is the maximum of the share price of the stock over the interval $[0, T]$ and τ_A is the first time that the share price of the stock reaches A.

   (c) Find the price of an option with payoff $e^{-\beta\tau_A}$, for some fixed β.